Kramers degeneracy

The energy levels of a system that contains an odd number of $S=1/2$ particles are at least doubly degenerated.
time-reversal symmetry

spin-less case

\( T=K \): conjugation operator

\[ K \Psi = \overline{\Psi} \]

\[ KK \Psi = K \overline{\Psi} = \Psi \]

\[ \langle K \Psi | K \phi \rangle = \langle \phi | \Psi \rangle \]

\[ Kp = -p \]

\[ Kv(r) = v(r) \]

**let us check the definition**

\[ i\hbar \frac{\partial \Psi(r,t)}{\partial t} = H \Psi(r,t) \]

\[ K \left( i\hbar \frac{\partial \Psi(r,t)}{\partial t} \right) = KH \Psi(r,t) \]

\[ -i\hbar \frac{\partial \overline{\Psi}(r,t)}{\partial t} = \overline{H \Psi(r,t)} \]

\[ i\hbar \frac{\partial \Psi(r,t)}{\partial (-t)} = H \overline{\Psi(r,t)} \]

\[ \overline{\Psi}(r,t) = K \Psi(r,t) = \Psi'(r,-t) \]
time-reversal symmetry

one electron with spin

\[ T = \sigma_y K \]
\[ \langle T\psi_\sigma | T\phi_{\sigma'} \rangle = \langle \phi_{\sigma'} | \psi_\sigma \rangle \]

\[ T\psi \chi_{\uparrow} = -i\overline{\psi} \chi_{\downarrow} \]
\[ TT\psi \chi_{\sigma} = -\psi \chi_{\sigma} \]

\[ T\mathbf{L} = -\mathbf{L} \quad T\mathbf{S} = -\mathbf{S} \]

let us check the definition

\[ i\hbar \frac{\partial \psi(r, t) \chi_\sigma}{\partial t} = H\psi(r, t) \chi_\sigma \]

\[ \hat{T} \left( i\hbar \frac{\partial \psi(r, t) \chi_\sigma}{\partial t} \right) = \hat{T} H\psi(r, t) \chi_\sigma \]

\[ i\hbar \frac{\partial (i\overline{\psi}(r, t) \chi_\downarrow)}{\partial (-t)} = H(i\overline{\psi}(r, t) \chi_\downarrow) \]
\[ i\hbar \frac{\partial (-i\overline{\psi}(r, t) \chi_\uparrow)}{\partial (-t)} = H(-i\overline{\psi}(r, t) \chi_\uparrow) \]
we can make time-even combinations of
\[ \Psi \quad \text{and} \quad \hat{T}\Psi \]
a time-even perturbation (distortion)
can remove the degeneracy

\[ T^2 = -1 \]
no time-even combinations
\[ \langle \Psi | \hat{T}\Psi \rangle = 0 \]
a time-even perturbation (distortion)
cannot remove the degeneracy

adding extra degeneracies in irreps.

no-spin case

<table>
<thead>
<tr>
<th>C(_3)</th>
<th>E</th>
<th>C(_3)</th>
<th>C(^2)</th>
<th>basis</th>
<th>rep. matrix</th>
<th>abelian group!</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>f</td>
<td>D</td>
<td>( \omega = e^{2\pi i/3} )</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>w</td>
<td>w(^2)</td>
<td>Tf</td>
<td>D(^*)</td>
<td>D and D(^*) are NOT equivalent</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>w(^2)</td>
<td>w</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

the two lower lines are typically given as 2-fold degenerate representation E because of time-reversal symmetry

why 2-fold degenerate? the two basis functions f and Tf are connected by time reversal operator! They form a 2-dimensional invariant linear space

Kramers degeneracy is not the only effect of time reversal
adding extra degeneracies in irr. reps.

no-spin case, T=K

\[ \{ \Psi^i_m \} \quad \text{and} \quad \{ T \Psi^i_m \} \quad \text{degenerate} \]

\[ O(g) \Psi^i_m = \sum_{m'} D_{m'm} (g) \Psi^i_{m'} \quad \text{first irr rep} \]

\[ O(g) T \Psi^i_m = TO(g) \Psi^i_m = \sum_{m'} [D_{m'm} (g)]^* T \Psi^i_{m'} \quad \text{second irr rep} \]

are these two irr reps equivalent or not?

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selection rules

experiment

theory ?
Fermi golden rule

transition probability from state $i$ to state $f$

$$T_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle \Psi_f | V | \Psi_i \rangle|^2 \rho(E_f - E_i)$$

$V$: interaction of system with external field

$i, f$ states of system

$G$: group of system
selection rules

\[ \langle \Psi^i_\mu | f | \Psi^j_\nu \rangle \]

- Identify irr. rep. decomposition of \( f \)
  - Vectors
  - Axial vectors
  - Tensors...

\[ \sum_{\lambda} a_\lambda D^\lambda \]

- Decompose the product of irr reps of \( f \) and the right vector into irr rep

\[ D^\lambda \otimes D^j = \bigoplus_l a_{l\lambda j} D_l \]

- Is there a \( l = i \)?

\[ D^i \otimes D^\lambda \otimes D^j = a_{A1} D^{A1} \oplus \ldots \]

\[ a_{A1} \neq 0? \]